hep-th/9706099, ROM2F–97–26, DFPD 97/TH/26

NON–HOLOMORPHIC TERMS IN N=2 SUSY WILSONIAN ACTIONS AND RG EQUATION

D. BELLISAI[1], F. FUCITO[2], M. MATONE[3], G. TRAVAGLINI[2,4]

[1] *Dipartimento di Fisica, Università di Roma "Tor Vergata"*
Via della Ricerca Scientifica, 00133 Roma, ITALY

[2] *I.N.F.N. – Sezione di Roma II "Tor Vergata", 00133 Roma, ITALY*

[3] *Dipartimento di Fisica "G. Galilei" – I.N.F.N. – Università di Padova*
Via Marzolo 8, 35131 Padova, ITALY

[4] *Dipartimento di Fisica, Università di Roma "La Sapienza"*
P.le Aldo Moro 5, 00185 Roma, ITALY

ABSTRACT

In this paper we first investigate the Ansatz of one of the present authors for $K(\Psi,\bar{\Psi})$, the adimensional modular invariant non–holomorphic correction to the Wilsonian effective Lagrangian of an $N=2$ globally supersymmetric gauge theory. The renormalisation group β–function of the theory crucially allows us to express $K(\Psi,\bar{\Psi})$ in a form that easily generalises to the case in which the theory is coupled to N_F hypermultiplets in the fundamental representation of the gauge group. This function satisfies an equation which should be viewed as a fully non–perturbative "non–chiral superconformal Ward identity". We also determine its renormalisation group equation. Furthermore, as a first step towards checking the validity of this Ansatz, we compute the contribution to $K(\Psi,\bar{\Psi})$ from instantons of winding number $k=1$ and $k=2$. As a by–product of our analysis we check a non–renormalisation theorem for $N_F=4$.

1 Introduction

In a celebrated paper, Seiberg and Witten studied a globally $N = 2$ supersymmetric Yang–Mills theory (SYM) with $SU(2)$ gauge group [1]. Subsequently they extended their analysis to theories with additional hypermultiplets (SQCD)[2]. They were able to exactly determine the Wilsonian effective action up to two derivatives and four fermions In terms of an $N = 2$ chiral superfield Ψ, it is proportional to a holomorphic function $\mathcal{F}(\Psi)$ called the prepotential. From a physical point of view, the Wilsonian effective action describes the low–energy degrees of freedom of the $N = 2$ microscopic supersymmetric theory. This achievement was possible thanks to a certain number of conjectures which were suggested by the physics of the problem. It was later shown in [3] that, in the case of $N = 2$ SYM, these assumptions follow from the symmetries of the theory and from the inversion formula first derived in [4] (subsequently generalised to SQCD in [5]), and are consistent with microscopic instanton computations in the cases of SYM and SQCD [6, 7, 8, 9, 10, 11, 12].

Since the moduli space of vacua of the theory is a thrice–punctured Riemann sphere, one can study the transformation properties of $\mathcal{F}(\Psi)$ under the modular group $\Gamma(2)$. The result of such an exercise is the inversion formula in [4], which relates $\mathcal{F}(\Psi)$ and its first derivative to a modular invariant function. The entire physical content of the theory can now be extracted from this differential equation [3, 4, 10, 12], which was also derived as an anomalous superconformal Ward identity in [13].

As it is well–known, a Wilsonian effective Lagrangian can be expanded in powers of the external momentum over some subtraction scale. Much in the same vein of the previous analysis, the investigation of the modular properties of the complete Wilsonian action leads to the conclusion that the term with four–derivatives/8–fermions, which we will denote by $K(\Psi, \bar{\Psi})$, is a modular invariant [14]. However, it seems that also the higher–order terms are modular invariant. Let us denote the non–holomorphic part of the Wilsonian effective action by $\hat{S}[\Psi, \bar{\Psi}]$; furthermore, let S, T be the $SL(2, \mathbb{Z})$

generators with $S^2 = 1$ and $(ST)^3 = 1$. In [14], it was shown that $\hat{S}[\Psi, \bar{\Psi}]$ does not transform under the action of T while, under duality, $\mathcal{F}(\Psi) \to \mathcal{F}_D(\Psi_D) = \mathcal{F}(\Psi) + \Psi_D \Psi$. Now if the action of T on $\hat{S}[\Psi, \bar{\Psi}]$ is trivial and the group has only two generators, the action of S must be trivial too, since

$$\hat{S}[\Psi, \bar{\Psi}] = (ST)^3 \circ \hat{S}[\Psi, \bar{\Psi}] = S^3 \circ \hat{S}[\Psi, \bar{\Psi}] = S \circ \hat{S}[\Psi, \bar{\Psi}] \quad . \tag{1.1}$$

However, we observe that the above modular invariance is considered with respect to the S and T action defined in [14] whereas, strictly speaking, a function $G(\Psi, \bar{\Psi})$ is said to be modular invariant if $G(\gamma(\Psi), \gamma(\bar{\Psi})) = G(\Psi, \bar{\Psi})$, $\gamma \in SL(2, \mathbb{Z})$.

Let us now leave this argument on the side and let us remark that the perturbative 1–loop term and the contribution of instantons of winding number $k = 1$ to $K(\Psi, \bar{\Psi})$ were computed in [15, 16]. On the basis of these results, and by using uniformisation theory, one of the present authors was able to write a modular invariant function which satisfies the constraints imposed by perturbative and instanton calculations and which has no other singularities but the one at weak coupling [17]. This function satisfies the physical requirements of the theory, for example it vanishes at those points of the moduli space where monopoles or dyons become massless: we consider it to be a candidate for the expression of $K(\Psi, \bar{\Psi})$. Its actual form will be reviewed in section 3 of this work, where we also write it in terms of the β–function of the theory, and find the renormalisation group equation satisfied by K. This function also an equation which should be viewed as a fully non–perturbative "non–chiral superconformal Ward identity". In that same section we also extend the Ansatz to the case of SQCD with N_F hypermultiplets. Furthermore, we study the higher–derivative corrections to the SYM and SQCD effective Lagrangians, and in particular the contributions of instantons of winding number $k = 1, 2$ to the real adimensional function $K(\Psi, \bar{\Psi})$. This is a first step in the direction of checking the proposal in [17] and that of section 3. As we will

discuss in section 4, the situation is more involved than in the case of the holomorphic part of the effective Lagrangian, and we cannot provide here a check for the expression of $K(\Psi, \bar{\Psi})$. We plan to come back on this point in a future publication.

The plan of the paper is the following: in section 2 we briefly review the solution of [1] to fix the notations and compute the relationship between the Pauli–Villars renormalisation group invariant scale and that appearing in [1]. We do this in great detail because we will need it in the following and because the literature is plagued with inconsistent notations. The content of section 3 has been discussed above. We start section 4 by computing the $k = 1$ contribution to $K(\Psi, \bar{\Psi})$. It turns out to be in agreement with the result of [16], which was derived by different methods. In the second part of the same section we compute the $k = 2$ contribution, for $N = 2$ SYM and SQCD. Furthermore, we check a recent result concerning a non–renormalisation theorem in the case of four flavours [18]. While we were writing this paper a work by Dorey *et al.* [19] has appeared in which computations partly similar to ours, in the case of winding number $k = 1$, are carried out and the non–renormalisation theorem for $N_F = 4$ is checked by using scaling arguments. Our results agree with theirs.

2 A Review of the Seiberg–Witten Model

The Lagrangian density for the microscopic $N = 2$ SYM theory, in the $N = 2$ supersymmetric formalism is given by

$$L = \frac{1}{16\pi} \mathrm{Im} \int d^2\theta d^2\tilde{\theta} \; \mathcal{F}(\Psi) \quad . \tag{2.1}$$

Ψ transforms in the adjoint representation of the gauge group G (which will be $SU(2)$ from now on). Re–expressing the Lagrangian density in the $N = 1$ formalism, we have

$$L = \frac{1}{16\pi} \mathrm{Im} \left[\int d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}, V) + \int d^2\theta f_{ab}(\Phi) W^a W^b \right] \quad , \tag{2.2}$$

where a, b are indices of the adjoint representation of G. The Kähler potential $K(\Phi, \bar{\Phi}, V)$ and the holomorphic function $f_{ab}(\Phi)$ are given, in terms of $\mathcal{F}$, by

$$K(\Phi, \bar{\Phi}, V) = (\bar{\Phi} e^{-2V})^a \frac{\partial \mathcal{F}}{\partial \Phi^a} \quad , \tag{2.3}$$

$$f_{ab}(\Phi) = \frac{\partial^2 \mathcal{F}}{\partial \Phi^a \partial \Phi^b} \quad . \tag{2.4}$$

The classical action for the $N = 2$ SYM theory is obtained by choosing for $\mathcal{F}$ the functional form

$$\mathcal{F}_{\text{cl}}(\Psi) = \frac{\tau_{\text{cl}}}{2} (\Psi^a \Psi^a) \quad , \tag{2.5}$$

where we conventionally define τ_{cl} as

$$\tau_{\text{cl}} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad . \tag{2.6}$$

Our normalisations are the same as in [1]. After eliminating the auxiliary fields, the classical action of the theory is given by

$$S = S_{\text{G}} + S_{\text{H}} + S_{\text{F}} + S_{\text{Y}} + S_{\text{pot}} \quad . \tag{2.7}$$

S_{G} is the usual gauge field action, the kinetic terms for the Fermi and Bose fields minimally coupled to the gauge field A_μ are

$$S_{\text{F}}[\lambda, \bar{\lambda}, A] = \int d^4x \; \bar{\lambda}^{\dot{A}a} \Big[\not{D}(A) \lambda_{\dot{A}} \Big]^a \quad , \tag{2.8}$$

where $\lambda_{\dot{A}}$ are the two gauginos, $\dot{A} = 1, 2$, and

$$S_{\text{H}}[\phi, \phi^\dagger, A] = \int d^4x \; (D\phi)^{\dagger a} (D\phi)^a \quad . \tag{2.9}$$

The Yukawa interactions are given by

$$S_{\text{Y}}[\phi, \phi^\dagger, \lambda, \bar{\lambda}] = \sqrt{2} g \epsilon^{abc} \int d^4x \; \phi^{a\dagger} (\lambda^b_{\dot{1}} \lambda^c_{\dot{2}}) \; + \; \text{h.c.} \tag{2.10}$$

and finally $S_{pot} = \int d^4x \; V(\phi, \phi^\dagger)$ comes from the potential term

$$V(\phi, \phi^\dagger) = \mathrm{Tr}[\phi, \phi^\dagger]^2 \quad , \tag{2.11}$$

for the complex scalar field. As required by supersymmetry, one has $V(\phi, \phi^\dagger) \geq 0$. The condition $V(\phi, \phi^\dagger) = 0$ implies that $[\phi, \phi^\dagger] = 0$: ϕ is then a normal operator, and can be diagonalised by a unitary matrix: that is, a colour rotation. The most general (supersymmetric) classical vacuum configuration is then

$$\phi_0 = a \left(\Omega \frac{\sigma_3}{2} \Omega^\dagger \right) \quad , \; a \in \mathbb{C} \quad , \quad \Omega \in SU(2) \quad . \tag{2.12}$$

When $a \neq 0$ the $SU(2)$ gauge symmetry is spontaneously broken to $U(1)$. The classical vacuum "degeneracy" for the $N = 2$ SYM theory is lifted neither by perturbative nor by non–perturbative quantum corrections [25, 26]. In fact any non–zero superpotential would explicitly break the extended supersymmetry of the model; however the Witten index of the theory is non–zero [27], so supersymmetry stays unbroken. We then have a full quantum moduli space, $\mathcal{M}_{SU(2)}$, for the low–energy theory. The effective Lagrangian for the massless $U(1)$ fields $\Phi = \left\{ \phi^3, \lambda^3_{\alpha \mathrm{i}}, F^3 \right\}$ will again be of the form

$$L_{eff} = \frac{1}{16\pi} \mathrm{Im} \left[\int d^2\theta \mathcal{F}''(\Phi) W W + \int d^2\theta d^2\bar{\theta} \bar{\Phi} \mathcal{F}'(\Phi) \right] \quad , \tag{2.13}$$

where $W = \left\{ A^3_\mu, \lambda^3_{\alpha \dot{2}}, D^3 \right\}$.

The low energy dynamics is then governed by a unique function $\mathcal{F}(\Phi)$, the effective prepotential, whose functional form is not restricted by supersymmetry. The crucial property of $\mathcal{F}(\Phi)$, first proved in [28], is holomorphicity. In analogy with (2.6) we can also define an effective coupling constant as

$$\tau(a) = \mathcal{F}''(a) \quad . \tag{2.14}$$

It is a simple exercise to rewrite (2.13) in the component field formalism.[1] This way

[1] Throughout the article we will use the conventions of Wess and Bagger [29] for the

we obtain

$$L_{eff} = \frac{1}{4\pi}\mathrm{Im}\Big[-\mathcal{F}''(\phi)\left(|\partial_\mu\phi|^2 + i\bar\lambda_{\dot A}\,\bar\partial\!\!\!/\lambda^{\dot A} + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}\right) + \frac{1}{\sqrt{2}}\mathcal{F}'''(\phi)\lambda_{\dot 1}\sigma^{\mu\nu}\lambda_{\dot 2}F_{\mu\nu} + \frac{1}{4}\mathcal{F}^{IV}(\phi)\lambda_{\dot 1}^2\lambda_{\dot 2}^2\Big] + \cdots \quad , \tag{2.15}$$

where the dots stand for terms of higher order in the coupling constant. The effective description of the low–energy dynamics in terms of the $U(1)$ superfields Φ, W is not appropriate for all vacuum configurations. In particular, the quantum moduli space $\mathcal{M}_{SU(2)}$ is better described in terms of the variable a and its dual $a_D = \partial_a\mathcal{F}$. When the gauge group is $SU(2)$, we can describe $\mathcal{M}_{SU(2)}$ in terms of the gauge–invariant coordinate $u =<\mathrm{Tr}\phi^2>$. Then $\mathcal{M}_{SU(2)}$ is the Riemann sphere with punctures at $u = \infty$ and, in the normalisation of [1], at $u = \pm\Lambda^2$.

At the classical level

$$\mathcal{F}_{\mathrm{cl}}(a) = \frac{\tau_{\mathrm{cl}}}{2}a^2 \quad , \tag{2.16}$$

however perturbative as well as non–perturbative effects modify the expression of the prepotential. We shall then write

$$\mathcal{F}(a) = \mathcal{F}_{pert}(a) + \mathcal{F}_{np}(a) \quad , \tag{2.17}$$

including the classical contribution in the first term. The perturbative contribution has been calculated by Seiberg [31] and is exactly determined thanks to the holomorphicity requirements on $\mathcal{F}(a)$ and to the $U(1)_R$ symmetry

$$U(1)_R: \quad \lambda_{\dot A} \longrightarrow e^{i\alpha}\lambda_{\dot A} \ , \quad \phi \longrightarrow e^{2i\alpha}\phi \ . \tag{2.18}$$

The associated current J^μ_R is anomalous

$$J_{\mu R} = \bar\lambda_{\dot 1}\bar\sigma_\mu\lambda_{\dot 1} + \bar\lambda_{\dot 2}\bar\sigma_\mu\lambda_{\dot 2} + 2i\phi^\dagger\overleftrightarrow{\partial}_\mu\phi \ , \ \partial_\mu J^\mu_R = -\frac{i}{32\pi^2}(F^a_{\mu\nu}\tilde F^a_{\mu\nu})(4N_c) \quad , \tag{2.19}$$

product of Weyl spinors and integration on superspace. We also define the Euclidean σ_μ, $\bar\sigma_\mu$ matrices as $\sigma_\mu = (\mathbb{1}, i\sigma^a)$, $\bar\sigma_\mu = (\mathbb{1}, -i\sigma^a)$, σ^a, $a = 1,2,3$ being the usual Pauli matrices, and the (anti)self-dual matrices $(\bar\sigma_{\mu\nu})\,\sigma_{\mu\nu}$ are $\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu) = i\eta^a_{\mu\nu}\sigma^a$, $\bar\sigma_{\mu\nu} = \frac{1}{2}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu) = i\bar\eta^a_{\mu\nu}\sigma^a$, where $\eta^a_{\mu\nu}, \bar\eta^a_{\mu\nu}$ are the 't Hooft symbols defined in [30].

(in our case the number of colours is taken to be $N_c = 2$). The discrete subgroup $\mathbb{Z}_8 \subset U(1)_R$, generated by the transformations (2.18) with $\alpha_m = (2\pi/8)m$, $m \in \mathbb{Z}$ is a symmetry of the full quantum theory, since in this case the action functional S transforms as

$$S \longrightarrow S + i8k\alpha_m = S + 2\pi i m \quad . \tag{2.20}$$

At a given point in the u–moduli space the $\mathbb{Z}_8$ symmetry spontaneously breaks down to $\mathbb{Z}_4$, since the $U(1)_R$ charge of u is $+4$. However, (2.20) tells us that the points u and $-u$ correspond to physically equivalent theories. We now immediately rewrite (2.18) in terms of the $U(1)$ superfield Ψ of the $N = 2$ supersymmetry as

$$U(1)_R : \quad \Psi(x, \theta) \longrightarrow \Psi'(x, \theta') = e^{2i\alpha}\Psi(x, \theta e^{-i\alpha}) \quad : \tag{2.21}$$

if we now assign a charge of $+1$ to θ, the charge of Ψ will be $+2$ in such a way that the classical prepotential (2.5) is invariant. Then the perturbative effective Lagrangian

$$L_{pert}[\Psi] = \frac{1}{16\pi}\mathrm{Im}\int d^2\theta d^2\tilde{\theta}\ \mathcal{F}_{pert}[\Psi(x, \theta)] \quad , \tag{2.22}$$

transforms in

$$\begin{aligned} L^{(\alpha)}_{pert}[\Psi'] \quad &= \quad \frac{1}{16\pi}\mathrm{Im}\int d^2\theta d^2\tilde{\theta}\ \mathcal{F}_{pert}[e^{2i\alpha}\Psi(x, \theta e^{-i\alpha})] = \\ & \quad \frac{1}{16\pi}\mathrm{Im}\int d^4\theta\ e^{-4i\alpha}\mathcal{F}_{pert}[e^{2i\alpha}\Psi(x, \theta)] \quad , \end{aligned} \tag{2.23}$$

where $d^4\theta = d^2\theta d^2\tilde{\theta}$. After a little algebra we get

$$L_{pert} + \delta_\alpha L_{pert} = \frac{1}{16\pi}\mathrm{Im}\int d^4\theta \left[1 + 4i\alpha\left(-1 + \Psi^2\frac{\partial}{\partial\Psi^2}\right)\right]\mathcal{F}_{pert}(\Psi) \ . \tag{2.24}$$

Furthermore, we know that under a $U(1)_R$ transformation

$$\delta_\alpha L_{pert} = -(4N_c\alpha)\left(\frac{1}{32\pi^2}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\right) \quad , \tag{2.25}$$

(with $N_c = 2$), so that

$$L_{pert} + \delta_\alpha L_{pert} = \frac{1}{16\pi} \mathrm{Im} \int d^4\theta \ \left[\mathcal{F}(\Psi) - \frac{2\alpha}{\pi} \Psi^2 \right] \quad , \tag{2.26}$$

from which it immediately follows that[2]

$$(\Psi^2 \frac{\partial}{\partial \Psi^2} - 1) \mathcal{F}_{pert}(\Psi) = \frac{i}{2\pi} \Psi^2 \quad . \tag{2.27}$$

This is the semiclassical version [13, 32, 33] of the non–perturbative relation

$$i\pi \left(\mathcal{F} - \frac{a}{2} \frac{\partial \mathcal{F}}{\partial a} \right) = < \mathrm{Tr}\ \phi^2 > \quad , \tag{2.28}$$

obtained in [4] and subsequently re–derived in [13]. The solution of (2.27) is

$$\mathcal{F}_{pert}(\Psi) = \frac{i}{2\pi} \Psi^2 \ln \frac{\Psi^2}{\mu^2} \quad , \tag{2.29}$$

where μ can be fixed by the value of the coupling constant at some subtraction point. The normalisation of the (one–loop) perturbative contribution must be fixed together with the non–perturbative contributions and the definition of the renormalisation group invariant (RGI) scale Λ. To this end we first write the non–perturbative prepotential as

$$\mathcal{F}_{np}(a) = \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a} \right)^{4k} a^2 \quad , \tag{2.30}$$

and similarly

$$u(a) = \frac{1}{2} a^2 + \sum_{k=1}^{\infty} \mathcal{G}_k \left(\frac{\Lambda}{a} \right)^{4k} a^2 \quad . \tag{2.31}$$

It easy to check that the expressions (2.30), (2.31) possess the correct invariance properties under the $\mathbb{Z}_8$ symmetry. The values of the $\mathcal{F}_k$'s and the $\mathcal{G}_k$'s are meaningful only if one specifies the choice of the RGI scale Λ, and can be obtained via a k–instanton calculation [6, 9, 12]. In the following we shall need the expressions for the 1–instanton contributions to $u(a)$, which has been found to be [6, 12]

$$< \mathrm{Tr}\phi^2 >_{k=1} = \frac{\Lambda_{PV}^4}{a^2} \quad . \tag{2.32}$$

[2] Disregarding terms which vanish when integrated in $d^4\theta$.

Here Λ_{PV} is the Pauli–Villars RGI invariant scale, which naturally arises when performing instanton calculations after the cancellation of the determinants of the kinetic operators of the various fields [34]. We shall fix in a moment its relationship with the scale employed in [1]. Note that the relation (2.28) gives the $\mathcal{F}_k$'s as a function of the $\mathcal{G}_k$'s,

$$2i\pi k\mathcal{F}_k = \mathcal{G}_k \quad . \tag{2.33}$$

By making some hypotheses on the structure of the moduli space and on the monodromies of τ around its singularities, Seiberg and Witten were able to obtain the expressions of $a(u)$ and $a_D(u)$, which are given by

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2} dx\, \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} \quad , \tag{2.34}$$

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{\Lambda^2}^{u} dx\, \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} \quad , \tag{2.35}$$

where Λ is the Seiberg–Witten RGI scale (to be matched against the Pauli–Villars one). We now put

$$a_D(u) = \frac{\sqrt{2u}}{\pi} g(1/u) \quad , \tag{2.36}$$

where

$$g(1/u) = \int_{\Lambda^2/u}^{1} dz\, \frac{\sqrt{z-1}}{\sqrt{z^2-\Lambda^4/u^2}} = \tag{2.37}$$

$$\int_{\Lambda^2/u}^{1} dz\, \left[\frac{\sqrt{z-1}}{\sqrt{z^2-\Lambda^4/u^2}} - \frac{i}{z}\right] + \int_{\Lambda^2/u}^{1} dz\, \frac{i}{z} \quad . \tag{2.38}$$

The perturbative constant ($u \gg \Lambda^2$) contribution to $g(1/u)$ is

$$\int_0^1 dz\, \left[\frac{\sqrt{z-1}}{z} - \frac{i}{z}\right] = 2i\ln\frac{2}{e} \quad , \tag{2.39}$$

so that

$$a_D(u) \to i\frac{\sqrt{2u}}{\pi}\ln\frac{4u}{(e\Lambda)^2} \quad . \tag{2.40}$$

Using the asymptotic expansion (2.31) we finally obtain an expression for a_D as a function of a in the perturbative regime,

$$a_D(a) \to \frac{i}{\pi} a \ln \frac{2a^2}{(e\Lambda)^2} \quad , \tag{2.41}$$

that is, in the same limit,

$$\mathcal{F}_{pert}(a) = \frac{i}{2\pi} a^2 \ln \frac{2a^2}{e^3\Lambda^2} \quad . \tag{2.42}$$

This sets the normalisation of the classical and perturbative contributions. From (2.42) it follows that

$$\tau_{pert}(a) = \mathcal{F}''_{pert}(a) = \frac{i}{\pi} \ln \frac{2a^2}{\Lambda^2} \quad . \tag{2.43}$$

We now examine the first instanton correction to $u(a)$; via the relation (2.33) we will then fix the normalisation of the $\mathcal{F}_k$'s. Expanding the expression (2.34) of the modulus a as a function of u for $u \gg \Lambda^2$ we get

$$\begin{aligned} a(u) &= \frac{\sqrt{2}}{\pi}\Big[\int_{-\Lambda^2}^{\Lambda^2} dx \, \frac{1}{\sqrt{\Lambda^4 - x^2}} - \frac{1}{8u^2}\int_{-\Lambda^2}^{\Lambda^2} dx \, \frac{x^2}{\sqrt{\Lambda^4 - x^2}} + \mathrm{O}(\Lambda^8/u^4)\Big] = \\ & \quad \sqrt{2u}\Big[1 - \frac{\Lambda^4}{16u^2} + \mathrm{O}(\Lambda^8/u^4)\Big] \quad . \end{aligned} \tag{2.44}$$

In the same approximation we also have that

$$u(a) = \frac{a^2}{2}\left[1 + 2\mathcal{G}_1 \left(\frac{\Lambda}{a}\right)^4 + \mathrm{O}\left(\frac{\Lambda}{a}\right)^8\right] \quad ; \tag{2.45}$$

substituting into (2.44) we get

$$a = a\left[1 + \mathcal{G}_1 \left(\frac{\Lambda}{a}\right)^4 + \mathrm{O}\left(\frac{\Lambda}{a}\right)^8\right] \cdot \left\{1 - \frac{1}{4}\left(\frac{\Lambda}{a}\right)^4 \left[1 + \mathrm{O}\left(\frac{\Lambda}{a}\right)^8\right]\right\} \quad , \tag{2.46}$$

and, for consistency, we must impose

$$\mathcal{G}_1 = \frac{1}{4} \quad , \tag{2.47}$$

with respect to the RGI scale Λ in [1]. Comparing (2.47) with (2.32) we find

$$\Lambda_{PV} = \frac{\Lambda}{\sqrt{2}} \quad . \tag{2.48}$$

The holomorphic prepotential $\mathcal{F}(a)$ is then given by

$$\begin{aligned}\mathcal{F}(a) \quad &= \quad \frac{i}{2\pi} a^2 \ln \frac{2a^2}{e^3 \Lambda^2} + a^2 \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} = \\ & \quad \frac{i}{2\pi} a^2 \ln \frac{a^2}{e^3 \Lambda_{PV}^2} + a^2 \sum_{k=1}^{\infty} \mathcal{F}_k \; 2^{2k} \left(\frac{\Lambda_{PV}}{a}\right)^{4k} \quad ,\end{aligned} \tag{2.49}$$

where $\mathcal{F}_1 = \mathcal{G}_1/2\pi i = 1/8\pi i$. Finally, when we add N_F hypermultiplets, the holomorphic prepotential $\mathcal{F}^{(N_F)}(a)$ becomes

$$\mathcal{F}^{(N_F)}(a) = \frac{i}{8\pi}(4-N_F) a^2 \ln \frac{a^2}{e^3 (\Lambda_{PV}^{(N_F)})^2} + a^2 \sum_{k=1}^{\infty} \mathcal{F}_k^{(N_F)} \; 2^{2k} \left(\frac{\Lambda_{PV}^{(N_F)}}{a}\right)^{k(4-N_F)} \quad , \tag{2.50}$$

where $\mathcal{F}_{2k+1}^{(N_F)}$=0 in the presence of massless hypermultiplets. This is a consequence of the $Z_{4(4-N_F)}$ chiral symmetry group of the full quantum theory [2].

3 Non–holomorphic corrections and the β–function

Let us briefly describe the general form of the higher–derivative corrections to the Lagrangian (2.2). Since an effective Lagrangian is written as an expansion in the space of momenta, the next–to–leading contributions will come out of the terms with four or more derivatives or eight or more fermions. In the case of $N = 2$ SYM theory, they will be written as a finite expansion in spinor derivatives,

$$S_{NL}(\Psi, \bar{\Psi}) = \int d^4x d^4\theta d^4\bar{\theta} \left[K(\Psi, \bar{\Psi}) + G(\Psi, \bar{\Psi}) D\Psi D\Psi \bar{D}\bar{\Psi}\bar{D}\bar{\Psi} + \cdots + \mathrm{O}(D^4, \bar{D}^4)\right] \quad . \tag{3.1}$$

If we assign the scaling dimension $[dx] = 1$, $[d\theta] = -1/2$ and $[D] = 1/2$, as a consequence of $N = 2$ supersymmetry the expansion will contain only terms with even

dimension. Furthermore, the $U(1)_R$ anomaly and the non–perturbative corrections are completely encoded in the analytic prepotential $\mathcal{F}$, which is the only holomorphic term that can appear in the effective Lagrangian. Therefore (3.1) is integrated over the whole superspace. From now on we will restrict our attention to the first term $K(\Psi, \bar{\Psi})$ in (3.1), which is adimensional and does not contain spinor derivatives of Ψ and $\bar{\Psi}$.

We now consider the derivation of K proposed in [17]. Let $H = \{w | \mathrm{Im}\, w > 0\}$ be the upper half plane endowed with the Poincaré metric $ds_P^2 = (\mathrm{Im}\, w)^{-2}|dw|^2$. Since $\tau = \partial_a^2 \mathcal{F}$ is the inverse of the map uniformising $\mathcal{M}_{SU(2)}$, it follows that the positive–definite metric

$$ds_P^2 = \frac{|\partial_a^3 \mathcal{F}|^2}{(\mathrm{Im}\, \tau)^2}|da|^2 = \frac{|\partial_u \tau|^2}{(\mathrm{Im}\, \tau)^2}|du|^2 = e^{\varphi}|du|^2 \quad , \tag{3.2}$$

is the Poincaré metric on $\mathcal{M}_{SU(2)}$. This implies that φ satisfies the Liouville equation

$$\partial_{\bar{u}} \partial_u \varphi = \frac{e^{\varphi}}{2} \quad . \tag{3.3}$$

Observe that this equation is satisfied since, for any fundamental domain F in H, $\tau(u)$ is a *univalent* (*i.e.* one–to–one) map between $\mathcal{M}_{SU(2)}$ and F. In this context we stress that $\tau(u)$ is not properly a function; rather it is a *polymorphic* function (*i.e.* it is Möbius transformed after going around non–trivial cycles). Therefore classical theorems concerning standard meromorphic functions do not hold. In particular, $\mathrm{Im}\, \tau(u)$ is a zero mode of the Laplacian. Observe that on the moduli space $\tau(u)$ is holomorphic as zeroes and poles are at the punctures (that is missing points). Zeroes and poles are manifest on the compactified moduli space. However, these critical points are absent in the case of higher genus Riemann surfaces without punctures. This follows from the fact that punctures correspond to points $\tau \in \mathbb{R} = \partial H$. In particular, as the fundamental domains of negatively curved Riemann surfaces without punctures Σ belong to H, it follows that in these cases τ is a holomorphic nowhere vanishing function on Σ. In

particular, $\Delta\ \mathrm{Im}\,\tau = 0$. In [4, 20, 3] it was shown how the results of [1] are naturally described in the framework of uniformisation theory. We now show how the function $K(a, \bar{a})$ derived in [17] naturally arises in this context.

To see this let us first recall some asymptotics for the Poincaré metric. Let us consider the Riemann sphere with elliptic or parabolic points (punctures) at $u_1, ..., u_{n-1}$, $u_n = \infty$. Near an elliptic point the behaviour of the Poincaré metric is

$$e^{\varphi} \sim \frac{4q_k^2 r_k^{2q_k-2}}{\left(1 - r_k^{2q_k}\right)^2} \quad , \tag{3.4}$$

where q_k^{-1} is the ramification index of u_k and $r_k = |u - u_k|$, $k = 1, \ldots, n-1$, $r_n = |u|$. Taking the $q_k \to 0$ limit we get the parabolic singularity (puncture)

$$e^{\varphi} \sim \frac{1}{r_k^2 \log^2 r_k} \quad . \tag{3.5}$$

It follows that in $\mathcal{M}_{SU(2)}$ case the Poincaré metric e^{φ} vanishes only at the puncture $u = \infty$, where $\varphi \sim -2\ln(|u| \ln |u|)$. Furthermore, e^{φ} is divergent only at the punctures $u = \pm\Lambda^2$, where $\varphi \sim -2\ln(|u \mp \Lambda^2| \ln |u \mp \Lambda^2|)$.

Let us now gather the known results on $K(\Psi, \bar{\Psi})$. First observe that in [15] it was proved that, to the one–loop order

$$K(\Psi, \bar{\Psi}) \sim c \ln \frac{\Psi}{\Lambda} \ln \frac{\bar{\Psi}}{\Lambda} \quad , \tag{3.6}$$

where c is a constant which was recently calculated [21] in the formalism of harmonic superspace for $0 \leq N_F \leq 4$. The non–holomorphic terms in the effective Lagrangian are $U(1)_R$–invariant. If we follow the reasoning made for $\mathcal{F}$ (which eventually led to (2.27)) we get, in particular,

$$\int d^4\theta d^4\bar{\theta} \left\{ \Psi \frac{\partial}{\partial \Psi} - \bar{\Psi} \frac{\partial}{\partial \bar{\Psi}} \right\} K(\Psi, \bar{\Psi}) = 0 \quad , \tag{3.7}$$

which should be considered as a semiclassical Ward identity for K. The solution of this equation is simply given, modulo Kähler transformations, by

$$K(y,\bar{y}) = P(y+\bar{y}) + y\bar{g}(\bar{y}) + \bar{y}g(y) \quad , \tag{3.8}$$

where $y = \ln \Psi/\Lambda$, g is an arbitrary function and $P = \bar{P}$.[3] In particular, the term found in [15] is a solution to this equation, but it seems that, in principle, no non–renormalisation theorem prevents us from considering solutions with higher order polynomials in $\ln(\Psi\bar{\Psi}/\Lambda^2)$. These terms would represent higher–loop contributions to K. However, in the case of SQCD with $N_F = 4$ massless hypermultiplets and gauge group $SU(2)$, we know that the β–function vanishes, so that no scale Λ arises in the theory. In this case the only possible function of Ψ/Λ which can appear in the solution (3.8) is a term linear in the product $\ln(\Psi/\Lambda)\ln(\bar{\Psi}/\Lambda)$ (or, up to purely chiral or antichiral terms, quadratic in $\ln(\Psi\bar{\Psi}/\Lambda^2)$) [18]; indeed, only in this case the scale Λ is a fake (it does not multiply non–holomorphic terms in the Lagrangian), as it should for a scale–invariant theory.

Let us go back to the $N_F = 0$ case. Besides (3.6) we know that that K is a modular invariant [14] and that the one–instanton contribution is [16][4]

$$K(\Psi,\bar{\Psi})|_{k=1} = \frac{1}{8\pi^2}\left(\frac{\Lambda}{\Psi}\right)^4 \ln\frac{\Psi}{\Lambda}\frac{\bar{\Psi}}{\Lambda} + \text{h.c.} \tag{3.9}$$

Strictly speaking, a function $G(\Psi,\bar{\Psi})$ is said to be modular invariant if $G(\gamma(\Psi),\gamma(\bar{\Psi})) = G(\Psi,\bar{\Psi})$, $\gamma \in SL(2,\mathbb{Z})$. However, $K(\Psi,\bar{\Psi})$ has the invariance $T \circ K(\Psi,\bar{\Psi}) = K(\Psi,\bar{\Psi})$ and $S \circ K(\Psi,\bar{\Psi}) = K(\Psi,\bar{\Psi})$. While in the former case there is no change in the functional structure of K, in the latter, according to the S–dual formulation of the theory, where $\mathcal{F}(\Psi)$ is replaced by $\mathcal{F}_D(\Psi_D)$, the function $S \circ K(\Psi,\bar{\Psi})$ should be constructed with the building block $\mathcal{F}_D(\Psi_D)$ (which replaces $\mathcal{F}(\Psi)$ in the construction of $K(\Psi,\bar{\Psi})$).

[3]It is a trivial exercise to show that (3.7) is completely equivalent to the superspace–integrated version of equation (3.7) of [21].

[4]From now on we will denote by Λ the Pauli–Villars RGI scale.

Let us discuss why $\mathcal{F}(\Psi)$ should be considered as a building block for $K(\Psi,\bar{\Psi})$. First of all, one can observe that the geometry determined by $\mathcal{F}$ is that of the Riemann sphere with three punctures. Then, by S–duality, modular invariance and general arguments, it is quite natural to believe that K should be a well–defined function on $\mathcal{M}_{SU(2)}$, that is a real "function" of $u,\bar{u}$. On the other hand the inversion formula (2.28) tells us that we can express u by means of $\mathcal{F}(\Psi)$. Therefore, $\mathcal{F}(\Psi)$ is the building block for $K(\Psi,\bar{\Psi})$. This is a useful result since, as we will see, it implies a differential equation for $K(\Psi,\bar{\Psi})$, which is the non–chiral analogue of (2.28). On the other hand (2.28), which is equivalent to a second–order equation, is actually a (anomalous) superconformal Ward identity [13]. Then, the equation we will get should be interpreted as a non–chiral superconformal Ward identity.

The request of modular invariance indicates that K should be constructed in terms of the geometrical building blocks of the thrice–punctured Riemann sphere $\mathcal{M}_{SU(2)}$. The comparison between the asymptotics (3.5) and (3.6) suggests that the Poincaré metric should have a rôle in defining K. In particular, we observe that, in order to be well–defined on the u–moduli space, the logarithmic terms should come out of a function which has to be globally defined. This would also respect the symmetries of the theory. The above analysis suggested the following proposal [17]:

$$K(\Psi,\bar{\Psi}) = \alpha\frac{e^{-\varphi(\mathcal{G}(\Psi),\overline{\mathcal{G}(\Psi)})}}{|\mathcal{G}^2(\Psi)-\Lambda^4|} \quad , \tag{3.10}$$

where α is a real constant to be determined via an explicit calculation, and

$$e^{\varphi(u,\bar{u})} = \frac{|\partial_u\tau|^2}{(\operatorname{Im}\tau)^2} \quad , \tag{3.11}$$

is the Poincaré metric on $\mathcal{M}_{SU(2)}$. The expression (3.10) can also be written in the form

$$K(\Psi,\bar{\Psi}) = 4\alpha\pi^2 e^{2\varphi_{SW}}|\mathcal{G}^2(\Psi)-\Lambda^4| \quad , \tag{3.12}$$

or

$$K(\Psi,\bar{\Psi}) = 2\alpha\pi e^{\varphi_{SW}(\mathcal{G}(\Psi),\overline{\mathcal{G}(\Psi)})-\frac{\varphi}{2}(\mathcal{G}(\Psi),\overline{\mathcal{G}(\Psi)})} \quad , \tag{3.13}$$

where

$$e^{\varphi_{SW}(u,\bar{u})} = |\partial_u a|^2 \mathrm{Im}\,\tau \quad , \tag{3.14}$$

is the Seiberg–Witten metric on $\mathcal{M}_{SU(2)}$.

Let us now consider the geometrical meaning of $K(\Psi,\bar{\Psi})$. According to (3.13) the $(1/2,1/2)$–differential K is proportional to the Seiberg–Witten metric times the inverse of the square root of the Poincaré metric. The interesting point is that the structure of (3.14) does not prevent us from considering for K a suitable modification of the Liouville equation which is satisfied by the Poincaré metric. In particular, looking at the structure of (3.14), it is easy to see that after a sufficient number of times one acts with the derivative operators, the effect of the Seiberg–Witten metric on the Liouville equation can be eliminated. In particular, setting

$$Y(a,\bar{a}) = K(a,\bar{a})\partial_a\partial_{\bar{a}} \ln K(a,\bar{a}) \quad , \tag{3.15}$$

we have the "non–chiral superconformal Ward identity"[5]

$$\partial_{\bar{a}}\partial_a \ln Y(a,\bar{a}) = 0 \quad . \tag{3.16}$$

3.1 $K(\Psi,\bar{\Psi})$ from the β–function

In [22] the renormalisation group equation (RGE) and the exact β–function was derived in the $SU(2)$ case. Also, similar structures have been considered in the framework of the Witten–Dijkgraaf–Verlinde–Verlinde equation in the $SU(3)$ case [23]. It would be interesting to understand the scaling properties of K. As it is constructed in terms of $\mathcal{F}$, one could imagine that the RGE for $\mathcal{F}$ should play a role. The RGE, derived in

[5] We thank Gaetano Bertoldi for interesting discussions on this equation.

[22], is

$$\partial_\Lambda \mathcal{F}(a,\Lambda) = \frac{\Lambda}{\Lambda_0}\partial_{\Lambda_0}\mathcal{F}(a_0,\Lambda_0)e^{-2\int_{\tau_0}^{\tau} dx\beta^{-1}(x)} \quad , \tag{3.17}$$

where

$$\beta(\tau) = \Lambda \left(\partial_\Lambda \tau\right)_u \quad , \tag{3.18}$$

is the β–function. Remarkably, the β–function admits a geometrical interpretation as the chiral block for the Poincaré metric, namely [22]

$$ds_P^2 = \left|\frac{\beta}{2u\,\mathrm{Im}\,\tau}\right|^2 |du|^2 = e^{\varphi}|du|^2 \quad . \tag{3.19}$$

On physical grounds it is clear that, the β–function should vanish at $u = 0$. However, this degeneracy should not appear in the relevant geometrical objects. Remarkably, this is actually the case. To be more precise, the above expression for K admits the equivalent general representation

$$K(a,\bar{a}) = 4\alpha\pi\frac{|\mathcal{G}(a)|(\mathrm{Im}\,\tau)^2}{|\beta||\partial_a\mathcal{G}(a)|^2} \quad . \tag{3.20}$$

3.2 The $1 \leq N_F \leq 4$ Case

As the above expression for K does not refer to a particular underlying geometry, we can consider (3.20) as a general model–independent expression for K. In particular, observe that its asymptotic expansion can be performed by just using the one for the prepotential $\mathcal{F}$. However, there is still another equivalent form for K which is particularly useful in order to perform asymptotic analyses. We have in mind the fact that, in the presence of massless hypermultiplets, only instantons with even k contribute. Then, in order to get a suitable expression for K, we introduce the function [22]

$$\beta^{(a)}(\tau) = \Lambda \left(\partial_\Lambda \tau\right)_a \quad , \tag{3.21}$$

whose relation with the β–function is [22]

$$\beta(\tau) = 2u\frac{\partial_u a}{a}\beta^{(a)}(\tau) \quad . \tag{3.22}$$

By (3.20) and (3.22) we have

$$K(a,\bar{a}) = 2\alpha\pi\frac{|a|(\operatorname{Im}\tau)^2}{|\beta^{(a)}||\partial_a\mathcal{G}(a)|} \quad . \tag{3.23}$$

To better illustrate the rôle of the β–function in the non–holomorphic contribution, we use a result in [22] where it was shown that

$$u = \Lambda^2 e^{-2\int_{\tau_0}^{\tau} dx\beta^{-1}(x)} \quad , \tag{3.24}$$

where $u(\tau_0) = \Lambda^2$ (in the $N_F = 0$ case, $\tau_0 = 0$). Then K has the form

$$K(a,\bar{a}) = \alpha\pi\left|\frac{a}{\Lambda}\right|^2 |F|^2 e^{\int_{\tau_0}^{\tau}\beta^{-1} + \overline{\int_{\tau_0}^{\tau}\beta^{-1}}}(\operatorname{Im}\tau)^2 \quad , \tag{3.25}$$

where

$$F(a,\bar{a}) = \frac{\beta^{1/2}}{\beta^{(a)}} \quad . \tag{3.26}$$

Thanks to (3.24) and (3.25) it follows that K satisfies the RGE

$$\Lambda\left(\partial_\Lambda K(a,\bar{a})\right)_{a,\bar{a}} = 2\left[\operatorname{Re}\left(\frac{\beta^{(a)}}{\beta} + \beta^{(a)}\partial_\tau \ln F\right) + \frac{\operatorname{Im}\beta^{(a)}}{\operatorname{Im}\tau} - 1\right]K(a,\bar{a}) \quad . \tag{3.27}$$

One can check that when only instantons with even k contribute to $\mathcal{F}$, then this would also be the case for the expression (3.23) for K.

Finally we note that in the $N_F = 4$ case the above construction breaks down. In particular, in this case the underlying geometry is trivial. As a consequence, the non–trivial global aspects of moduli spaces, which actually generate non–perturbative corrections, do not arise for $N_F = 4$. This is already clear for the chiral part $\mathcal{F}$ which is proportional to a^2. As in general the function K is built in terms of $\mathcal{F}$, we see that there is no way to get non–holomorphic contributions to K but the one–loop term, whose structure has a global meaning since the underlying geometry is trivial.

4 Non–perturbative contributions to $K(\Psi,\bar\Psi)$

Let us now discuss the series expansion for $K(\Psi,\bar\Psi)$ in the case of SYM theory. We can rewrite (3.10) as

$$K(\Psi,\bar\Psi) = \frac{64\alpha}{\pi^2}\frac{|\mathcal{G}^2(\Psi)-4\Lambda^4|(\mathrm{Im}\,\tau(\Psi))^2}{|\Psi|^4|\hat\tau(\Psi)-\tau(\Psi)|^4} \quad , \tag{4.1}$$

where

$$\hat\tau(\Psi) = \frac{1}{\Psi}\frac{\partial\mathcal{F}}{\partial\Psi} \quad . \tag{4.2}$$

It can be fixed by using the result in [21]; however, this is not enough to get a complete check of the validity of (3.20) and (3.23), as we will discuss in the following.

Expanding (4.1) up to the order relevant to 2–instanton calculations, and neglecting purely chiral or antichiral terms, we find

$$\begin{aligned} K(x,\bar x) \;\simeq\; & \alpha\Big\{\ln x\ln\bar x + x^4(3\ln\bar x - 2\ln^2\bar x - 2\ln\bar x\ln x) + \\ & x^8\left(-\frac{21}{2}\ln x\ln\bar x - \frac{21}{2}\ln^2\bar x + \frac{57}{8}\ln\bar x\right) + \\ & x^4\bar x^4\left(\frac{9}{4} - 6\ln\bar x + 2\ln^2\bar x + 4\ln x\ln\bar x\right) + \text{h.c.}\Big\} \quad , \end{aligned} \tag{4.3}$$

where $x = \Lambda/\Psi$.

Let us briefly comment on the functional dependence of the various terms appearing in the expansion. The first logarithmic term represents the one–loop perturbative contribution to $K(\Psi,\bar\Psi)$ which was first derived in [15]; it is to be noted that there are no higher–order (higher–loop) logarithmic corrections to $K(\Psi,\bar\Psi)$. This seems to be confirmed by recent results found in [24], where the two–loop correction to the effective Lagrangian (2.1) is shown to vanish. As far as the terms $x^{4k}\ln\bar x$ are concerned, they appear explicitly in the k–instanton calculations, while the terms with $x^{4k}\ln\bar x\ln x$ and $x^{4k}\ln^2\bar x$ are expected to be one–loop corrections around the k–instanton configuration.

As a matter of fact, in this case there are no constraints coming from holomorphicity requirements and from the anomalous $U(1)_R$ symmetry which forbid the existence of loop corrections around instanton configurations [31]. Finally, the terms $x^{4m}\bar{x}^{4n}$ and logarithmic corrections are expected to represent m–instanton/n–antiinstanton contributions and loop corrections around this configuration. In the case of SQCD this situation is simply modified in the presence of massless hypermultiplets, since the expansion contains only non–perturbative contributions from m–instanton/n–antiinstanton where m, n are even numbers and one–loop corrections around these configurations. In the sequel we will perform 1– and 2–instanton calculations which will give contributions to $K(\Psi, \bar{\Psi})$ of the form expected from the conjecture in [17]. Let us now make a remark which will become clear after the instanton computation will be performed. If we differentiate $K(x, \bar{x})$ twice with respect to x and twice with respect to $\bar{x}$ (to obtain $K_{xx\bar{x}\bar{x}}$), the terms containing the $\ln x\bar{x}$ and the $\ln^2 x\bar{x}$ give contributions which sum. Therefore, for an unambiguous check of the conjectures (3.20), (3.23), one needs not only 1–instanton or 2–instanton but also mixed 1–instanton/1-antiinstanton results and perturbative corrections around all the aforementioned configurations. Anyway, as a first step towards the check these proposals, we now compute the non–perturbative (1–instanton and 2–instanton) contributions to $K(\Psi, \bar{\Psi})$.

In terms of $N = 1$ superspace the four–derivative term reads [14]:

$$\frac{1}{16}\int d^2\theta d^2\bar{\theta}\Big[K_{\phi\bar{\phi}}(\Phi, \bar{\Phi})(D^\alpha D_\alpha \Phi \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi} + 2\bar{D}_{\dot{\alpha}} D^\alpha \Phi D_\alpha \bar{D}^{\dot{\alpha}} \bar{\Phi} + 4 D^\alpha W_\alpha \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} -$$

$$4 D^{(\alpha} W^{\beta)} D_{(\alpha} W_{\beta)} - 4\bar{D}_{(\dot{\alpha}} \bar{W}_{\dot{\beta})} \bar{D}^{(\dot{\alpha}} \bar{W}^{\dot{\beta})} - 2 D^\alpha D_\alpha (W^\beta W_\beta) - 2\bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} (\bar{W}_{\dot{\beta}} \bar{W}^{\dot{\beta}})) -$$

$$2 K_{\phi\phi\bar{\phi}}(\Phi, \bar{\Phi}) W^\alpha W_\alpha D^\beta D_\beta \Phi - 2 K_{\phi\bar{\phi}\bar{\phi}}(\Phi, \bar{\Phi}) \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \bar{D}_{\dot{\beta}} \bar{D}^{\dot{\beta}} \bar{\Phi} +$$

$$K_{\phi\phi\bar{\phi}\bar{\phi}}(\Phi, \bar{\Phi}) \left(-8 W^\alpha D_\alpha \Phi \bar{W}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi} + 4 W^\alpha W_\alpha \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}\right)\Big] \quad , \tag{4.4}$$

where $K_\phi = \partial K / \partial \phi$. When written in the x–space this Lagrangian contains a four–field

strength vertex which is the one we will focus our attention on in our calculations:

$$\frac{1}{4}\int d^4x d^2\theta d^2\bar{\theta}\ K_{\phi\phi\bar{\phi}\bar{\phi}}(\Phi,\bar{\Phi})W^\alpha W_\alpha \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} =$$

$$\frac{1}{256}K_{aa\bar{a}\bar{a}}(a,\bar{a})\int d^4x\ \mathrm{Tr}(\sigma^{ab}\sigma^{cd})\mathrm{Tr}(\bar{\sigma}^{ef}\bar{\sigma}^{gh})F_{ab}F_{cd}F_{ef}F_{gh}\ . \tag{4.5}$$

Thus, the correlator we intend to study is

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle\ \ . \tag{4.6}$$

4.1 The $k=1$ Semiclassical Computation

The relevant configuration which contributes to this Green function is dictated by the sweeping–out procedure at the next–to–leading–order of [9]:

$$F_{\mu\nu} = F^{(0)}_{\mu\nu} + i\xi(x)\sigma_{[\nu}D_{\mu]}\bar{\lambda}^{(0)} + 2i\bar{\lambda}^{(0)}\bar{\sigma}_{\mu\nu}\bar{\varepsilon} + 2ig\xi^2(x)\bar{\lambda}^{(0)}\bar{\sigma}_{\mu\nu}\bar{\lambda}^{(0)}\ \ , \tag{4.7}$$

where $F^{(0)}_{\mu\nu}$ satisfies the equation

$$D^\mu F^{(0)}_{\mu\nu} = -2ig[\phi^\dagger_{\mathrm{cl}}, D_\nu\phi_{\mathrm{cl}}]\ \ , \tag{4.8}$$

with

$$\phi_{\mathrm{cl}} = \frac{x^2}{x^2+\rho^2}a^c(\sigma^c/2)\ \ , \tag{4.9}$$

and

$$\xi(x) = \xi + \rho^{-1}x^\mu\sigma_\mu\bar{\varepsilon}\ \ , \tag{4.10}$$

$$\bar{\lambda}^{(0)} = -i\sqrt{2}\xi'\,\not{D}\phi^\dagger_{\mathrm{cl}}\ \ . \tag{4.11}$$

Here, for simplicity, x stands for $x-x_0$, where x_0 is the centre of the 1–instanton configuration, and ρ is its size; finally $\bar{\eta}'^a_{\mu\nu} = R_{ab}\bar{\eta}^a_{\mu\nu}$, where R_{ab} is an $SU(2)$ rotation matrix which corresponds to global colour rotations.

We start by rederiving the result of [16] for the 1–instanton case in a different way. In the case $k=1$, the $N=2$ SYM measure on the moduli space is simply [30, 35]

$$\int d^3\Theta d^4x_0 \frac{d\rho}{\rho^5}\frac{2^7\pi^6}{g^8}(\mu\rho)^8\left(\frac{16\pi^2\mu}{g^2}\right)^{-2} d^2\xi d^2\xi' \left(\frac{32\pi^2\rho^2\mu}{g^2}\right)^{-2} d^2\bar{\varepsilon} d^2\bar{\varepsilon}' \exp(-S_{inst}) \quad , \tag{4.12}$$

where S_{inst} is the sum of the classical action, the Higgs and the Yukawa terms, and Θ^a, $a=1,2,3$ denotes the moduli associated with global colour rotations.

We observe first that $F_{\mu\nu}$ does not contain the superconformal collective coordinate $\bar{\varepsilon}'$ so that the integration over the superconformal fermionic coordinates must be completely saturated by the Yukawa action and we can ignore the terms in $F_{\mu\nu}$ which depend on the fermionic coordinate $\bar{\varepsilon}$. Therefore in evaluating the correlator (4.6), only the first, the second and the fourth term in the r.h.s. of (4.7) will be of interest. To lowest order in $g^2\rho^2|a|^2$, $F^{(0)}_{\mu\nu}$ becomes

$$F^{\rm cl}_{\mu\nu} = \frac{4\rho^2}{g}\frac{1}{x^2(x^2+\rho^2)^2}(-x^2\bar{\eta}'^a_{\mu\nu} + 2x_\lambda x_\nu \bar{\eta}'^a_{\mu\lambda} + 2x_\lambda x_\mu \bar{\eta}'^a_{\lambda\nu})\frac{\sigma^a}{2} \quad , \tag{4.13}$$

and the term proportional to ξ^2 is negligible. Then, in order to saturate the integration over the supersymmetric collective coordinates ξ, ξ', the product in (4.6) boils down to

$$F^{\rm cl}_{\mu\nu}(x_1)F^{\rm cl}_{\rho\sigma}(x_2)[i\xi\sigma_{[\nu}D_{\mu]}\bar{\lambda}^{(0)}(x_3)][i\xi\sigma_{[\nu}D_{\mu]}\bar{\lambda}^{(0)}(x_4)] \quad . \tag{4.14}$$

Now we have to extrapolate the relevant long–distance effective $U(1)$ fields (4.13):

$$F^{(3)\rm cl,LD}_{\mu\nu}(x) = \frac{4\rho^2}{g}\cdot\frac{1}{x^6}(-x^2\bar{\eta}'^3_{\mu\nu} + 2x_\lambda x_\nu \bar{\eta}'^3_{\mu\lambda} + 2x_\lambda x_\mu \bar{\eta}'^3_{\lambda\nu}) \quad , \tag{4.15}$$

and

$$i\xi\sigma_{[\nu}\partial_{\mu]}\bar{\lambda}^{(0)}_{LD} \quad , \tag{4.16}$$

where $\bar{\lambda}^{(0)}_{LD} = -i\sqrt{2}\xi' \not{\partial}\phi^{\dagger LD}_{\rm cl}$ and the suffix LD stands for long–distance. In this limit the covariant derivative becomes a simple one. In [9] a nice relationship between the

scalar Higgs field and the Higgs action in the long–distance limit was derived,

$$\phi^{\dagger}_{\mathrm{cl,LD}} = \bar{a} - a^{-1} S_H G(x, x_0) \quad , \tag{4.17}$$

where $G(x,x_0) = 1/4\pi^2(x-x_0)^2$ is the massless scalar propagator. As a consequence of this observation it is possible to recast (4.16) into the form

$$\frac{\sqrt{2}}{2}\frac{\partial}{\partial a} S_H \xi \sigma^{ab} \xi' G_{\mu\nu,ab}(x, x_0) \quad , \tag{4.18}$$

where $G_{\mu\nu,ab}(x,x_0)$ is the gauge–invariant propagator of the $U(1)$ field strength

$$G_{\mu\nu,ab}(x,x_0) = (\delta_{\nu b}\partial_\mu\partial_a - \delta_{\nu a}\partial_\mu\partial_b - \delta_{\mu b}\partial_\nu\partial_a + \delta_{\mu a}\partial_\nu\partial_b) G(x,x_0) \quad . \tag{4.19}$$

The integration on the superconformal collective coordinates, which are lifted in the background of the constrained instanton, is completely saturated by the Yukawa action S_Y, and one gets [36]

$$\int d^2\bar{\varepsilon} d^2\bar{\varepsilon}' \exp(-S_Y) = -2^9\pi^4 g^{-2}\rho^4\bar{a}^2 \quad . \tag{4.20}$$

The key observation is that the only dependence on the coordinates Θ is due to the insertion of $F^{(3)\mathrm{cl}}_{\mu\nu}$ and that, in the long–distance limit,

$$\begin{aligned} \int_{SU(2)/Z_2} d^3\Theta F^{(3)\mathrm{cl}}_{\mu\nu}(x_1) F^{(3)\mathrm{cl}}_{\rho\sigma}(x_2) &= \frac{8\pi^2}{3} F^{a\,\mathrm{cl}}_{\mu\nu}(x_1) F^{a\,\mathrm{cl}}_{\rho\sigma}(x_2) = \\ &= -\frac{16\pi^6\rho^4}{3g^2}\mathrm{Tr}(\bar{\sigma}^{ef}\bar{\sigma}^{gh}) G_{\mu\nu,ef}(x_1,x_0) G_{\rho\sigma,gh}(x_2,x_0) \ . \end{aligned} \tag{4.21}$$

Taking into account the other two insertions which saturate the integration over ξ, ξ' we finally obtain

$$\begin{aligned} \langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=1} = -\frac{15}{64\pi^2}\frac{\Lambda^4}{g^4\bar{a}^2a^6}\int d^4x_0 \mathrm{Tr}(\sigma^{ab}\sigma^{cd}) \\ G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\bar{\sigma}^{ef}\bar{\sigma}^{gh})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0) \quad . \end{aligned} \tag{4.22}$$

On the other hand the computation of the four–field strength vertex making use of the effective Lagrangian yelds

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{L-eff} = \frac{3}{32}K_{aa\bar{a}\bar{a}}(a,\bar{a})\int d^4x \mathrm{Tr}(\sigma^{ab}\sigma^{cd})$$

$$G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\bar{\sigma}^{ef}\bar{\sigma}^{gh})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0) \quad , \tag{4.23}$$

which finally reproduces the result [16]

$$K(a,\bar{a}) = \frac{1}{8\pi^2 g^4}\frac{\Lambda^4}{a^4}\ln\bar{a} \quad . \tag{4.24}$$

We can rewrite the 1–instanton correlator in a form which is well–suited to the generalisation to SQCD with $1 \leq N_F \leq 4$ massive hypermultiplets in the fundamental representation of $SU(2)$ (in the case in which at least one hypermultiplet is massless the non–perturbative contributions are expected to come only from m–instanton n–antiinstanton configurations where m, n are even),

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=1} = \frac{\pi^4}{2}\Big(\int d^4x_0 \mathrm{Tr}(\sigma^{ab}\sigma^{cd})G_{\mu\nu,ab}(x_1,x_0)$$

$$G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\bar{\sigma}^{ef}\bar{\sigma}^{gh})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0)\Big)\frac{\partial^2}{\partial a^2}\left[\int d\tilde{\mu}_1\rho^4\right] \quad , \tag{4.25}$$

where $d\tilde{\mu}_1$ is the "reduced" instanton measure obtained by extracting from the full measure the integration over the bosonic and fermionic translational coordinates [9, 12]. This formula generalises immediately by exchanging $d\tilde{\mu}_1$ with $d\tilde{\mu}_1^{N_F}$ [9], where

$$\int d\tilde{\mu}_1^{N_F} = -\frac{1}{16\pi^2 g^4}\frac{\Lambda_{N_F}^{4-N_F}}{a^2}\prod_{i=1}^{N_F} m_i \quad , \tag{4.26}$$

and m_i is the mass of the i–th hypermultiplet. By doing this we obtain

$$K(a,\bar{a})|_{N_F} = \frac{1}{8\pi^2 g^4}\frac{\Lambda_{N_F}^{4-N_F}}{a^2}\ln\bar{a}\prod_{i=1}^{N_F} m_i \quad , \tag{4.27}$$

which is in complete agreement with one of the results obtained in [37]. It is to be noted that in the case $N_F = 4$ the β-function vanishes identically so that the scale Λ_{N_F} must be replaced by $q = \exp(2i\pi\tau_{\mathrm{cl}})$, where τ_{cl} is defined in (2.14).

4.2 The $k = 2$ Computation

Let us now describe the calculation of the 2–instanton contribution to the real function $K(\Psi, \bar{\Psi})$. Again, the Green function which we are going to study is the simplest one, the four–field strength one. We will then be able to immediately generalise our calculation to the case of SQCD and to check the validity of the non–renormalisation theorem in the case $N_F = 4$ found in [18].

We start by briefly recalling how to determine gauge field configurations for a generic winding number k. The instanton field can be conveniently written in terms of the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction [38, 39]. To find an instanton solution of winding number k, one introduces a $(k+1) \times k$ quaternionic matrix

$$\Delta = a + bx \quad , \tag{4.28}$$

where x denotes a point of the one–dimensional quaternionic space $\mathbb{H} \equiv \mathbb{C}^2 \equiv \mathbb{R}^4$, $x = x^\mu \sigma_\mu$.[6] The gauge connection is then written in the form

$$A_\mu^{cl} = U^\dagger \partial_\mu U \quad , \tag{4.29}$$

where U is a $(k+1) \times 1$ matrix of quaternions providing an orthonormal frame of $\mathrm{Ker}\Delta^\dagger$, *i.e.*

$$\Delta^\dagger U = 0 \quad , \tag{4.30}$$

$$U^\dagger U = 1\!\!1_2 \quad . \tag{4.31}$$

The constraint (4.31) ensures that A_μ^{cl} is an element of the Lie algebra of the $SU(2)$ gauge group. The condition of self–duality on the field strength of (4.29) is imposed by restricting the matrix Δ to obey

$$\Delta^\dagger \Delta = f^{-1} \otimes 1\!\!1_2 \quad , \tag{4.32}$$

[6] We use the conventions of [12].

with f an invertible hermitian $k \times k$ matrix (of real numbers). The reparametrisation invariances of the ADHM construction [40] can be used to simplify the expressions of a and b. Exploiting this fact, in the following we will choose the matrix b to be

$$b = -\begin{pmatrix} 0_{1\times k} \\ 1\!\!1_{k\times k} \end{pmatrix} \quad . \tag{4.33}$$

From (4.29), one can compute the field strength of the gauge field, which reads

$$F^{\rm cl}_{\mu\nu} = 2U^{\dagger} b \sigma_{\mu\nu} f b^{\dagger} U \quad . \tag{4.34}$$

In the so–called singular gauge, one has

$$\begin{aligned} U_0 &= \sigma_0 \left(1 - \frac{1}{2} f_{lm} {\rm tr} v_l \bar{v}_m\right)^{1/2} \quad , \\ U_p &= -\frac{1}{|U_0|^2} \Delta_{pl} f_{lm} \bar{v}_m U_0 \quad , \end{aligned} \tag{4.35}$$

where $v_p = \Delta_{0p}$ and $l, m, p = 1, \ldots, k$. In the following we will need only the long–distance limit of these functions,

$$\begin{aligned} &\Delta_{pl} \sim b_{pl} x \quad , \qquad f_{lm} \sim \frac{1}{x^2} \delta_{lm} \quad , \\ &U_k \sim -\frac{1}{x^2} x \bar{v}_k U_0 \quad , \qquad U_0 \sim \sigma_0 \quad , \\ &\Delta_{0l} \sim 0 \quad . \end{aligned} \tag{4.36}$$

When $k = 2$ the most general instanton configuration can be written starting from the ADHM matrix

$$a = \begin{pmatrix} v_1 & v_2 \\ x_0 + e & d \\ d & x_0 - e \end{pmatrix} \quad . \tag{4.37}$$

Here

$$d = \frac{e}{4|e|^2} (\bar{v}_2 v_1 - \bar{v}_1 v_2) \quad , \tag{4.38}$$

as a consequence of the ADHM defining equations [41].

The fermionic zero–modes $\lambda^{(0)}_{\beta\dot{A}}$ are easily deduced from the gauge field zero–modes [40]

$$Z_\mu = U^\dagger C \bar{\sigma}_\mu f b^\dagger U - U^\dagger b f \sigma_\mu C^\dagger U \quad , \tag{4.39}$$

by recalling that, due to $N = 2$ SUSY,

$$\lambda^{(0)}_{\beta\dot{A}} = \sigma^\mu_{\beta\dot{A}} Z_\mu \quad , \tag{4.40}$$

($\dot{A} = 1, 2$ labels the two SUSY charges and $\beta = 1, 2$ is a spin index). For (4.39) to be transverse zero–modes, the $(k+1) \times k$ matrix C (for a generic instanton number k) must satisfy

$$\Delta^\dagger C = (\Delta^\dagger C)^T \quad , \tag{4.41}$$

where the superscript T stands for transposition of the quaternionic elements of the matrix (without transposing the quaternions themselves). The number of C's satisfying (4.41) is $8k$ [40]. We also need the form of the matrix C appearing in (4.40), which is constrained by (4.41) to describe the zero–modes of the $N = 2$ gauginos $\lambda^{(0)}_{\beta\dot{A}}$. To parallel the form of (4.37), we shall put

$$C_{\dot{1}} = \begin{pmatrix} \mu_1 & \mu_2 \\ 4\xi + \eta & \delta \\ \delta & 4\xi - \eta \end{pmatrix} \quad , \tag{4.42}$$

$$C_{\dot{2}} = \begin{pmatrix} \nu_1 & \nu_2 \\ 4\xi' + \eta' & \delta' \\ \delta' & 4\xi' - \eta' \end{pmatrix} \quad , \tag{4.43}$$

where δ, δ' are constrained by (4.41) to be

$$\begin{aligned} \delta &= \frac{e}{2|e|^2}(2\bar{d}\eta + \bar{v}_2\mu_1 - \bar{v}_1\mu_2) \quad , \\ \delta' &= \frac{e}{2|e|^2}(2\bar{d}\eta' + \bar{v}_2\nu_1 - \bar{v}_1\nu_2) \quad . \end{aligned} \tag{4.44}$$

In the long–distance limit, the 2–instanton field strength factorises in

$$F^{\mathrm{cl}\,LD}_{\mu\nu} = \frac{2}{x^6}[v_1\bar{x}\sigma_{\mu\nu}x\bar{v}_1 + (v_1 \to v_2)] = \frac{1}{x^6}[v_1(-x^2\bar{\sigma}_{\mu\nu} + 2x^\rho x_\mu\bar{\sigma}_{\rho\nu} + 2x^\rho x_\nu\bar{\sigma}_{\mu\rho})\bar{v}_1 + (v_1 \to v_2)] \quad . \tag{4.45}$$

On the other hand in [9] it was proved that, thanks to the geometrical properties of the ADHM construction, the relationship between the ξ, ξ' bilinear part in (4.7) and the Higgs action continues to hold for every winding number.

We start with the $k = 2$ $N = 2$ supersymmetric measure, which reads

$$\frac{1}{\mathcal{S}_2}\int d^4x_0 d^4e d^4v_1 d^4v_2 d^2\xi d^2\xi' d^2\eta d^2\eta' d^2\mu_1 d^2\mu_2 d^2\nu_1 d^2\nu_2 \exp(-S_{inst})\left(\frac{J_B}{J_F}\right)^{1/2} \quad . \tag{4.46}$$

$\mathcal{S}_2$ is the $k = 2$ symmetry factor which eliminates all the redundant copies of each field configuration which appears in the ADHM formalism [40, 9], and $J_B(J_F)$ is the Jacobian of the change of variables for the bosonic (fermionic) degrees of freedom. As in the calculation of the 2–instanton contribution to the $N = 2$ prepotential [9], we find it convenient to define the four combinations of the bosonic parameters:

$$\begin{aligned} L &= |v_1|^2 + |v_2|^2 \quad , \\ H &= L + 4|d|^2 + 4|e|^2 \quad , \\ \Omega &= v_1\bar{v}_2 - v_2\bar{v}_1 \quad , \\ \omega &= \frac{1}{2}\mathrm{tr}\ \Omega A_{00} \quad , \end{aligned} \tag{4.47}$$

where $A_{00} = \frac{i}{2}a^c\sigma^c$. In terms of these new variables it is possible to write the Higgs action as

$$S_H = 16\pi^2\left(L|A_{00}|^2 - \frac{|\omega|^2}{H}\right) = 4\pi^2|a|^2\left(L - \frac{|\Omega|^2\cos^2\theta}{H}\right) \quad , \tag{4.48}$$

and the Yukawa action as

$$S_{\mathrm{Y}} = 4\sqrt{2}\pi^2\Big[-\nu_k\bar{A}_{00}\mu_k + (\bar{\omega}/H)(\mu_1\nu_2 - \nu_1\mu_2 + 2\eta\delta' - 2\eta'\delta)\Big] \quad , \tag{4.49}$$

where $|\omega| = \frac{1}{2}|\Omega||a||\cos\theta|$ defines the polar angle θ. Finally

$$\frac{1}{\mathcal{S}_2}\left(\frac{J_B}{J_F}\right)^{1/2}\exp(-S_{\mathrm{cl}}) = 2^6\pi^{-8}\Lambda^8\frac{\big||e|^2-|d|^2\big|}{H} \quad . \tag{4.50}$$

As in the 1–instanton case the integration over the non–supersymmetric fermionic coordinates is saturated by the Yukawa action which gives

$$\begin{aligned}\int d^2\eta d^2\eta' d^2\mu_1 d^2\mu_2 d^2\nu_1 d^2\nu_2 \exp(-S_{\mathrm{Y}}) \quad &= \quad -\frac{2^5\pi^6\bar{a}^6\cos^2\theta}{|e|^4H'^2}L^2\left[\left(1+\frac{\cos^2\theta}{H'}\right)^2+\right.\\ &\left.\frac{1-|\Omega'|^2}{H'^2}\sin^2\theta\cos^2\theta\right] \quad ,\end{aligned} \tag{4.51}$$

where we have redefined $\Omega' = \Omega/L, H' = H/L$. The integration over the variable e is traded for the integration on H, *i.e.*

$$\int d^4e\,\frac{\big||e|^2-|d|^2\big|}{|e|^4} \quad \longrightarrow \quad \frac{\pi^2}{2}\int_{L+2|\Omega|}^{\infty} dH \quad . \tag{4.52}$$

As far as the two insertions of $F_{\mu\nu}$ bilinear in ξ, ξ' are concerned, it is possible to use a trick already exploited in the 1–instanton case. It consists in writing them as a second derivative of the instanton measure with respect to a [9]; the remaining two insertions, however, will have to be inserted and integrated explicitly. First of all let us write v_2 as a function of v_1, Ω, L,

$$v_2 = \left(\frac{\bar{\Omega}}{2} + \sqrt{|v_1|^2(L-|v_1|^2) - \frac{|\Omega|^2}{4}}\right)\frac{v_1}{|v_1|^2} \quad , \tag{4.53}$$

and insert this form in the long–distance limit of the 2–instanton classical configuration. The integration measure over v_1, L, Ω is written as

$$2\int_0^\infty dL\int_{|\Omega|\leq L} d^3\Omega\int_{L_-}^{L_+} d|v_1|^2\frac{1}{32\sqrt{(L_+-|v_1|^2)(|v_1|^2-L_-)}}\int_{S^3} d^3\Theta \quad , \tag{4.54}$$

where $\int_{S^3} d^3\Theta = 2\pi^2$ is the integration over the global colour rotations of the first centre of the instanton and $L_\pm = \frac{1}{2}(L \pm \sqrt{L^2 - |\Omega|^2})$. On the other hand

$$\int d^3\Omega = L^3 \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \int_0^1 |\Omega'|^2 d|\Omega'| \quad , \tag{4.55}$$

where θ is the angle between Ω and the direction singled out by the vev of the Higgs field. Again, as in the 1–instanton case, the key observation is that, in the long–distance limit,

$$\int d^3\Theta F^{3\,\mathrm{cl}}_{\mu\nu}(x) F^{3\,\mathrm{cl}}_{\rho\sigma}(y) = \frac{2\pi^2}{3} F^{a\,\mathrm{cl}}_{\mu\nu}(x) F^{a\,\mathrm{cl}}_{\rho\sigma}(y) = \tag{4.56}$$

$$-\frac{2\pi^6}{3} \mathrm{Tr}(\bar\sigma^{ab}\bar\sigma^{cd}) G_{\mu\nu,ab}(x,x_0) G_{\rho\sigma,cd}(y,x_0) \left(L|v_1|^2 - \frac{|\Omega|^2}{2}\sin^2\theta \right) \quad .$$

Putting everything together one obtains the following integral for the correlator:

$$\int d^4x_0 \int_0^1 d|\Omega'||\Omega'|^6 \int_{-1}^{1} d(\cos\theta)\cos^2\theta \int_{1+2|\Omega'|}^{\infty} \frac{dH'}{H'^3} \int_0^\infty dL L^7$$

$$[1 - |\Omega'|^2 \sin^2\theta](-4\pi^{14})\Lambda^8 \bar a^6 \left[\left(1 + \frac{\cos^2\theta}{H'}\right)^2 + \frac{1-|\Omega'|^2}{H'^2}\sin^2\theta\cos^2\theta \right]$$

$$\mathrm{Tr}(\bar\sigma^{ab}\bar\sigma^{cd}) G_{\mu\nu,ab}(x_1,x_0) G_{\rho\sigma,cd}(x_2,x_0) \mathrm{Tr}(\sigma^{ef}\sigma^{gh}) G_{\lambda\tau,ef}(x_3,x_0) G_{\kappa\theta,gh}(x_4,x_0)$$

$$\frac{\partial^2}{\partial a^2} \exp\left[-4\pi^2 L|a|^2 \left(1 - \frac{|\Omega'|^2\cos^2\theta}{H'}\right)\right] \quad , \tag{4.57}$$

and, after a trivial integration on L we get

$$\int d^4x_0 \int_0^1 d|\Omega'||\Omega'|^6 \int_{-1}^{1} d(\cos\theta)\cos^2\theta \int_{1+2|\Omega'|}^{\infty} \frac{dH'}{H'^3} [1 - |\Omega'|^2\sin^2\theta]$$

$$\left(-\frac{5\cdot 3^4 \cdot 7}{2^6\pi^2}\right) \frac{\Lambda^8}{\bar a^2 a^{10}} \frac{\left(1+\frac{\cos^2\theta}{H'}\right)^2 + \frac{1-|\Omega'|^2}{H'^2}\sin^2\theta\cos^2\theta}{\left(1 - \frac{|\Omega'|^2\cos^2\theta}{H'}\right)^8} \mathrm{Tr}(\bar\sigma^{ab}\bar\sigma^{cd}) G_{\mu\nu,ab}(x_1,x_0)$$

$$G_{\rho\sigma,cd}(x_2,x_0) \mathrm{Tr}(\sigma^{ef}\sigma^{gh}) G_{\lambda\tau,ef}(x_3,x_0) G_{\kappa\theta,gh}(x_4,x_0) \quad . \tag{4.58}$$

The remaining integrations over the adimensional variables $|\Omega'|, \cos\theta, H'$ can be easily performed by using a standard algebraic manipulation routine and give $1/42$. The final

result is then, restoring the explicit g dependence,

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=2} = -\frac{5\cdot 3^3}{2^7\pi^2 g^8}\frac{\Lambda^8}{\bar{a}^2 a^{10}}\int d^4x_0 \mathrm{Tr}(\bar{\sigma}^{ab}\bar{\sigma}^{cd})$$

$$G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\sigma^{ef}\sigma^{gh})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0) \quad . \tag{4.59}$$

Comparing this result to that of the effective Lagrangian gives

$$K(a,\bar{a})|_{k=2} = \frac{5}{32\pi^2 g^8}\frac{\Lambda^8}{a^8}\ln\bar{a} \quad , \tag{4.60}$$

which is our prediction for the 2–instanton contribution to the real function $K(\Psi,\bar{\Psi})$ while the 2–antiinstanton configuration contribution to K is simply the complex conjugate of (4.60).

Let us generalise our result to the case of $N_F \leq 4$ massless hypermultiplets which receives the first non–perturbative contribution from the 2–instanton sector and verify the non–renormalisation theorem of [18] for $N_F = 4$. As in the 1–instanton case (see (4.25)) it is possible to rewrite the four–field strength correlator as a double derivative of the "reduced" measure with respect to a

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=2} = \frac{\pi^4}{4}\frac{\partial^2}{\partial a^2}\left[\int d\tilde{\mu}_2\left(|v_1|^2 L - \frac{|\Omega|^2}{2}\sin^2\theta\right)\right] \tag{4.61}$$

$$\int d^4x_0 \mathrm{Tr}(\bar{\sigma}^{ab}\bar{\sigma}^{cd})G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\sigma^{ef}\sigma^{gh})G_{\lambda\tau,gh}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0) \quad ,$$

and the extension to the case $N_F > 0$ is performed by substituting the "reduced" measure $d\tilde{\mu}_2$ with $d\tilde{\mu}_2^{N_F}$ as defined in [11]:

$$\begin{aligned}\int d\tilde{\mu}_2^{N_F} \quad &= \quad -2^9\pi^7\bar{a}^2\Lambda_{N_F}^{(4-N_F)}\int_0^1 d|\Omega||\Omega|^2\int_{-1}^1 d(\cos\theta)\int_{1+2|\Omega|}^\infty \frac{dH}{H^3}\int_{S^3} d^3\Theta\int_0^\infty dLL \\ &\int_{L_-}^{L_+}\frac{d|v_1|^2}{\sqrt{(L_+-|v_1|^2)(|v_1|^2-L_-)}}\exp\left[-4\pi^2 L|a|^2\left(1-\frac{|\Omega|^2\cos^2\theta}{H}\right)\right]\cdot \\ &\cdot\sum_{n=0}^{N_F}\frac{M_{N_F-n}^{(N_F)}}{\pi^{4n}}\left.\frac{\partial^{2n}G}{\partial Z^{2n}}\right|_{Z=0} \quad .\end{aligned} \tag{4.62}$$

We have dropped for simplicity the primes on H, Ω; $G(Z)$ contains the contribution from the integration measure over the hypermultiplets and has the form

$$\begin{aligned}
G(Z) &= \left(\bar{\omega}L + \frac{iZ}{8\sqrt{2}}\right)^2 \left[\frac{\bar{a}^2}{16}|\Omega|^2 L^2 + \frac{L}{2H}\bar{a}\bar{\omega}L\left(\bar{\omega}L + \frac{iZ}{8\sqrt{2}}\right) + \frac{1-|\Omega|^2\sin^2\theta}{4H^2}\cdot\right. \\
&\quad \left.\cdot\left(\bar{\omega}L + \frac{iZ}{8\sqrt{2}}\right)^2\right] \exp\left[\frac{i\pi^2 Z}{\sqrt{2}H}|\Omega| aL\cos\theta\right] \quad . \qquad (4.63)
\end{aligned}$$

The $M^{(N_F)}_{N_F-n}$ are a set of $SO(2N_F)$ invariant polynomials in the masses m_n of the hypermultiplets:

$$\begin{aligned}
M^{(N_F)}_0 &= 1 \quad , \\
M^{(N_F)}_1 &= \sum_{n=1}^{N_F} m_n^2 \quad , \\
M^{(N_F)}_2 &= \sum_{n<p}^{N_F} m_n^2 m_p^2 \quad , \qquad (4.64) \\
\vdots \quad & \quad \vdots \\
M^{(N_F)}_{N_F} &= \prod_{n=1}^{N_F} m_n^2 \quad .
\end{aligned}$$

In the case of massless hypermultiplets, the only contribution to the correlator will come from the term with the $2N_F$–th derivative of $G(Z)$ and, writing the generic contribution to $K(\Psi, \bar{\Psi})$ as

$$K(\Psi,\bar{\Psi})|_{N_F<4} = K_2^{(N_F)} \frac{1}{\pi^2 g^8}\left(\frac{\Lambda_{N_F}}{\Psi}\right)^{2(4-N_F)} \ln\bar{\Psi} \quad , \qquad (4.65)$$

we find

$$\begin{aligned}
K_2^{(0)} &= \frac{5}{32} \quad , \qquad K_2^{(1)} = -\frac{3^3}{2^{10}} \quad , \\
K_2^{(2)} &= \frac{3}{2^{10}} \quad , \qquad K_2^{(3)} = -\frac{1}{2^{12}} \quad . \qquad (4.66)
\end{aligned}$$

For the case $N_F = 4$ we get

$$K(\Psi, \bar{\Psi})|_{N_F=4} = \frac{q^2}{3^3 2^{11} \pi^2 g^8} \ln \bar{\Psi} \quad , \tag{4.67}$$

which is a purely antichiral term. When integrated over the whole superspace it does not contribute to the effective action; this confirms thus the non–renormalisation theorem of [18]. In the case in which there are massive hypermultiplets, (4.65), (4.67) generalises immediately to the formula

$$K(\Psi, \bar{\Psi})|_{N_F} = \sum_{n=0}^{N_F} M_{N_F-n}^{(N_F)} K_2^{(N_F)} \frac{1}{\pi^2 g^8} \left(\frac{\Lambda_{N_F}}{\Psi}\right)^{2(4-N_F)} \ln \bar{\Psi} \quad , \tag{4.68}$$

provided that one replaces $\Lambda_{N_F}^{2(4-N_F)}$ with q^2 when $N_F = 4$. In this case the non–renormalisation theorem of [18], as already noted in [19], is spoilt by the presence of other energy scales represented by the masses of the hypermultiplets.

We observe that our investigation is stricly related to the "non–chiral" analogue of the Picard–Fuchs equations [7, 42] and the related integrable structure [45]. Also, the approach deserves to be generalised to the higher–rank group case [43, 44] and to the strong coupling region [46]. Finally, we observe that much of the theory seems related to Duistermaat–Heckman theorem [47]. In this context we observe that in a recent paper McArthur and Gargett a "Gaussian approach" to supersymmetric effective actions has been investigated [48].

Acknowledgements

D. B. would like to thank Marc Grisaru for discussions on the papers [15, 24]. F. F. was partly supported by the European Commission Contract CHRX–CT93–0340 and M. M. by the European Commission TMR programme ERBFMRX–CT96–0045.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; *ibid.* **B430** (1994) 485.

[2] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484.

[3] G. Bonelli, M. Matone and M. Tonin, Phys. Rev. **D55** (1997) 6466.

[4] M. Matone, Phys. Lett. **357B** (1995) 342.

[5] J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. **B367** (1996) 145; T. Eguchi and S.-K. Yang, Mod. Phys. Lett. **A11** (1996) 131.

[6] D. Finnell and P. Pouliot, Nucl. Phys. **B453** (1995) 225.

[7] K. Ito and N. Sasakura, Phys. Lett. **B382** (1996) 95.

[8] T. Harano and M. Sato, Nucl. Phys. **B484** (1997) 167; H. Aoyama, T. Harano, M. Sato and S. Wada, Phys.Lett. **B388** (1996) 331; N. Dorey, V. V. Khoze and M. P. Mattis, Phys. Lett. **B388** (1996) 324.

[9] N. Dorey, V. V. Khoze and M. P. Mattis, Phys. Rev. **D54** (1996) 2921.

[10] N. Dorey, V. V. Khoze and M. P. Mattis, Phys. Lett. **B390** (1997) 205.

[11] N. Dorey, V. V. Khoze and M. P. Mattis, Phys. Rev. **D54** (1996) 7832.

[12] F. Fucito and G. Travaglini, Phys. Rev. **D55** (1997) 1099.

[13] P. S. Howe and P. C. West, Nucl. Phys. **B486** (1996) 425.

[14] M. Henningson, Nucl. Phys. **B458** (1996) 445.

[15] B. de Wit, M. T. Grisaru and M. Roček, Phys. Lett. **B374** (1996) 297.

[16] A. Yung, Nucl. Phys. **B485** (1997) 38.

[17] M. Matone, Phys. Rev. Lett. **78** (1997) 1412.

[18] M. Dine and N. Seiberg, *Comment on Higher Derivative Operators in Some SUSY Field Theories*, SCIIP 97/12, RU–97–13, `hep-th/9705057`.

[19] N. Dorey, V. V. Khoze, M. P. Mattis, M. J. Slater and W. A. Weir, *Instantons, Higher Derivative Terms and Nonrenormalization Theorems in Supersymmetric Gauge Theories*, `hep-th/9706007`.

[20] M. Matone, Phys. Rev. **D53** (1996) 7354.

[21] S. V. Ketov, *On the next–to–leading–order correction to the effective action in N=2 gauge theories*, DESY 97–103, ITP–UH–18/97, `hep-th 9706079`.

[22] G. Bonelli and M. Matone, Phys. Rev. Lett. **76** (1996) 4107.

[23] G. Bonelli and M. Matone, Phys. Rev. Lett. **77** (1996) 4712.

[24] A. De Giovanni, M. T. Grisaru, M. Roček, R. von Unge and D. Zanon, *The N=2 Super Yang–Mills Low–Energy Effective Action at Two Loops*, IFUM568–FT, `hep-th/9706013`.

[25] N. Seiberg, Phys. Lett. **B328** (1993) 469.

[26] C. Gomez and R. Hernandez, *Electric–magnetic duality and effective field theories*, Madrid preprint FTUAM 96/36, `hep-th/9510023`.

[27] E. Witten, Nucl. Phys. **B202** (1982) 253.

[28] S. J. Gates, Jr. Nucl. Phys. **B238** (1984) 349.

[29] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, 1983.

[30] G. 't Hooft, Phys. Rev. **D14** (1976) 3432.

[31] N. Seiberg, Phys. Lett. **206B** (1988) 75.

[32] G. Travaglini, Ph. D. Thesis, Università di Roma "La Sapienza", 1996, unpublished.

[33] L. Alvarez–Gaumé and S. F. Hassan, *Introduction to S–Duality in $N = 2$ Supersymmetric Gauge Theories (A Pedagogical Review of the Work of Seiberg and Witten)* CERN–TH/96–371, `hep-th/9701069`.

[34] A. D'Adda and P. Di Vecchia, Phys. Lett. **73B** (1978) 162.

[35] C. Bernard, Phys. Rev. **D19** (1979) 3013.

[36] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. **B223** (1983) 445.

[37] A. Yung, *Higher Derivative Terms in the Effective Action of N=2 SUSY QCD from Instantons*, `hep-th/9705181`, PNPI–2168-TH–25.

[38] M. Atiyah, V. Drinfeld, N. Hitchin and Yu. Manin, Phys. Lett. **65A** (1978) 185.

[39] M. Atiyah, *Geometry of Yang–Mills Fields*, Lezioni Fermiane, Accademia Nazionale dei Lincei e Scuola Normale Superiore, Pisa (1979).

[40] H. Osborn, Ann. Phys. **135** (1981) 373.

[41] N. Christ, N. Stanton and E. Weinberg, Phys. Rev. **D18** (1978) 2013.

[42] A. Ceresole, R. D'Auria and S. Ferrara, Phys. Lett. **B339** (1994) 71; J. M. Isidro, A. Mukherjee, J. P. Nunes and H. J. Schnitzer, Nucl. Phys. **B492** (1997) 647; `hep-th/9703176`; `hep-th/9704174`.

[43] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. **B344** (1995) 169; P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. **74** (1995) 3931.

[44] I. M. Krichever and D. H. Phong, *On the Integrable Geometry of Soliton Equations and $N = 2$ Supersymmetric Gauge Theories*, `hep-th/9604199`; E. D'Hoker, I. M. Krichever and D. H. Phong, Nucl. Phys. **B489** (1997) 179; Nucl. Phys. **B489** (1997) 211; *The Renormalization Group Equation in $N = 2$ Supersymmetric Gauge Theories*, UCLA–96–TEP–40, `hep-th/9610156`; E. D'Hoker, Phys. Lett. **B397** (1997) 94.

[45] A. Gorsky, I. Krichever, A. Marshakov, A. Morozov and A. Mironov, Phys. Lett. **B355** (1995) 466; R. Donagi and E. Witten, Nucl. Phys. **B460** (1996) 299; E. Martinec and N. P. Warner, Nucl. Phys. **B459** (1996) 97; E. Martinec, Phys. Lett. **B367** (1996) 91; T. Nakatsu and K. Takasaki, Mod. Phys. Lett. **A11** (1996) 157; Int. J. Mod. Phys. **A11** (1996) 5505; `hep-th/9603219`; H. Itoyama and A. Morozov, Nucl. Phys. **B477** (1996) 855; **B491** (1997) 529; `hep-th/9601168`; R. Carroll, `solv-int/9511009`; `solv-int/9606005`.

[46] F. Ferrari and A. Bilal, Nucl. Phys. **B469** (1996) 387; A. Bilal and F. Ferrari, Nucl. Phys. **B480** (1996) 589.

[47] J. J. Duistermaat and G. J. Heckman, Invent. Math. **69** (1982) 259; R. F. Picken, J. Math. Phys. **31** (1990) 616.

[48] I. N. McArthur and T. D. Gargett, *A "Gaussian" Approach to Computing Supersymmetric Effective Actions*, LMU–TPW–97–9, `hep-th/9705200`.